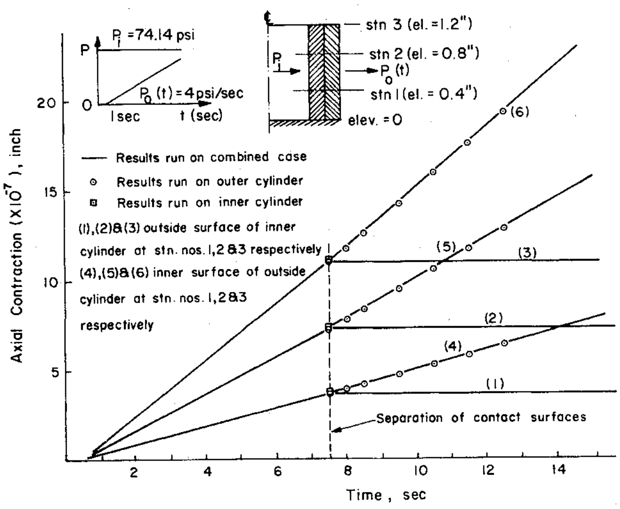

Fig. 2 Axial displacement at contact surfaces of two concentric cylinders.

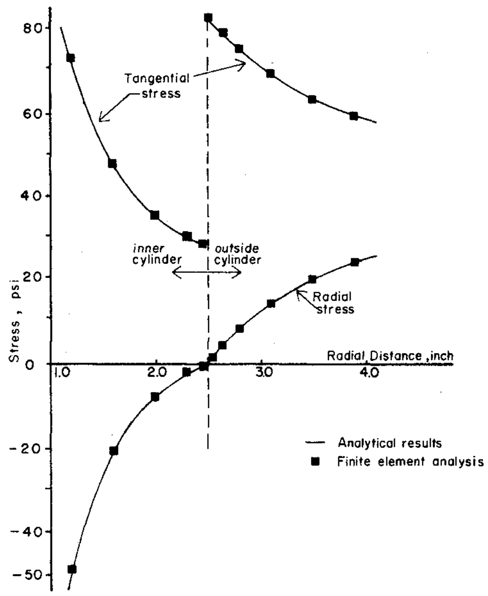

Fig. 3 Distribution of stresses in compound cylinder after separation.

from the end constraint. The various stress components in the cylinder system at separation are shown in Fig. 3.

The previous demonstration shows that in order to transmit pressure between two deforming solids in contact, the present method uses the idea of noncompressibility as exhibited by perfectly plastic interface elements capable of maintaining a hydrostatic condition in the interface. The same interface elements may also handle the separation of these solids with the appropriate switching of the element material properties from perfect plasticity to a low modulus of elasticity as indicated in Table 1. This technique is simple in principle and easy to apply. Its feasibility and applicability have been demonstrated with appropriate examples.

Another major advantage of this technique is its flexibility and potential for handling thermomechanical and inelastic contact problems. As mentioned earlier, the complicated thermomechanical contact behavior of a typical multi-material nuclear reactor fuel element has been satisfactorily simulated.[6]

The present method is limited to the frictionless sliding of contacting surfaces because of the lack of friction factors for practical structures. Such forces could be accommodated in the present scheme for example by varying interface element plasticity properties. The plastic behavior of this material should be related to the coefficient of friction of the contacting solids. This procedure would require considerable effort in formulation but with the unique advantages already indicated above, the effort could well be worthwhile.

Table 1 Material properties

	Inner cylinder	Outside cylinder	Interface layer: Before separation	Interface layer: After separation
Modulus of elasticity (E), psi	10^7	30×10^6	10^7	10^{-3}
Shear modulus of elasticity (G), psi	3.76×10^6	11.28×10^6	3.76×10^6	3.76×10^{-4}
Poisson's ratio	0.33	0.33	0.49	0.01
Yield strength, psi	30,000	90,000	1.0	10^7
Plastic modulus (E'), psi	0	0	0	0

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Static Deflection of Beams Subjected to Elastic Rotational Restraints

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Introduction

THE beam shown in Fig. 1 is simply supported and elastically restrained against rotation at both ends. The deflection shape caused by a general case of loading may be obtained by successively integrating the differential equation

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of equilibrium and satisfying the appropriate boundary conditions. The solution is a fourth order polynomial of three variables, namely, the rotational restraint parameters and the coordinate location along the beam span.

The purpose of this Note is to demonstrate that the deflection of the rotationally restrained beam can be expressed explicitly in terms of the rotational restraint constants and the deflection functions of the beam when subjected to limiting end-support boundary conditions, i.e., pinned, clamped, or combinations thereof.

Analysis

As shown in Fig. 1 the beam is assumed to be subjected to a general case of lateral loading consisting of a point load Q, distributed load $q(x)$, and a couple T. The end rotational restraints are represented by torsional springs of constants k_1 and k_2. The constants may assume any values between the limits 0 and ∞ which correspond to the pinned and clamped supports respectively. The moment M at the ends will be related to the angular rotation θ by

$$M_{12}^1 = k_1\ \theta_{12}^1 = \beta_1 EI/L\ \theta_{12}^1 \tag{1a}$$

$$M_{12}^2 = k_2\ \theta_{12}^2 = \beta_2 EI/L\ \theta_{12}^2 \tag{1b}$$

where β_1 and β_2 are nondimensional spring constants defined as $\beta_1 = k_1 L/EI$ and $\beta_2 = k_2 L/EI$, subscripts 1 and 2 indicate general values of β_1 and β_2, and superscripts 1 or 2 indicate the respective ends of the beam. The beam deflection $y_{12}(x)$ can be expressed in terms of the deflection shapes, $y_{00}(x)$, $y_{\infty 0}(x)$, $y_{0\infty}(x)$, and $y_{\infty\infty}(x)$, which correspond to the various combinations of the limiting end-support conditions, as

$$y_{12}(x) = C_1 y_{00}(x) + C_2 y_{\infty 0}(x) + C_3 y_{0\infty}(x) + C_4 y_{\infty\infty}(x) \tag{2}$$

where C_1, C_2, C_3, and C_4 are weighting factors, which are function of β_1 and β_2 only, and satisfy the equation

$$C_1 + C_2 + C_3 + C_4 = 1 \tag{3}$$

Differentiating Eq. (2) with respect to x we obtain the slope

$$\theta_{12}(x) = C_1\ \theta_{00}(x) + C_2\ \theta_{\infty 0}(x) + C_3\ \theta_{0\infty}(x) + C_4\ \theta_{\infty\infty}(x) \tag{4}$$

Thus at ends 1 and 2 the slope becomes

$$\theta_{12}^1 = C_1\ \theta_{00}^1 + C_3\ \theta_{0\infty}^1 \tag{5a}$$

$$\theta_{12}^2 = C_1\ \theta_{00}^2 + C_2\ \theta_{\infty 0}^2 \tag{5b}$$

The bending moment distribution on the beam is equivalent to that acting on a simply supported beam under the same external load plus end moments of magnitudes M_{12}^1 and M_{12}^2. Thus, using the principle of superposition, the slopes at the ends will be

$$\theta_{12}^1 = \theta_{00}^1 - M_{12}^1 L/3EI - M_{12}^2 L/6EI \tag{6a}$$

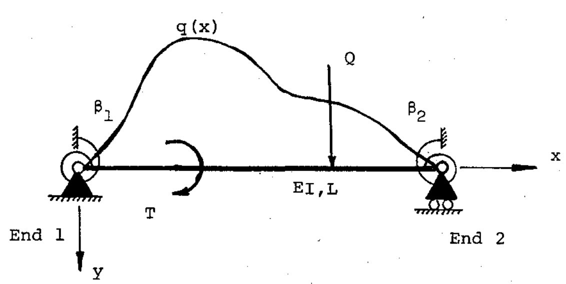

Fig. 1 Beam with elastic rotational restraints.

$$\theta_{12}^2 = \theta_{00}^2 - M_{12}^1 L/6EI - M_{12}^2 L/3EI \tag{6b}$$

substituting for M_{12}^1 and M_{12}^2 from Eq. (1) and rearranging the terms we get

$$\theta_{12}^1 \left(\frac{3+\beta_1}{3} \right) + \theta_{12}^2 \frac{\beta_2}{6} = \theta_{00}^1 \tag{7}$$

$$\theta_{12}^1 \frac{\beta_1}{6} + \theta_{12}^2 \left(\frac{3+\beta_2}{3} \right) = \theta_{00}^2 \tag{8}$$

solving for θ_{12}^1 and θ_{12}^2

$$\theta_{12}^1 = 2[2\ \theta_{00}^1 (3+\beta_2) - \theta_{00}^2\ \beta_2]/D \tag{9}$$

$$\theta_{12}^2 = 2[2\ \theta_{00}^2 (3+\beta_1) - \theta_{00}^1\ \beta_1]/D \tag{10}$$

where

$$D = 12 + 4(\beta_1 + \beta_2) + \beta_1\beta_2 \tag{11}$$

For the limiting case of $\beta_1 = 0$ and $\beta_2 = \infty$, Eq. (9) gives

$$\theta_{0\infty}^1 = (2\ \theta_{00}^1 - \theta_{00}^2)/2 \tag{12}$$

or

$$\theta_{00}^2 = 2(\theta_{00}^1 - \theta_{0\infty}^1) \tag{13}$$

Similarly for $\beta_1 = \infty$ and $\beta_2 = 0$, Eq. (10) yields

$$\theta_{00}^1 = 2(\theta_{00}^2 - \theta_{\infty 0}^2) \tag{14}$$

Substituting for θ_{00}^2 and θ_{00}^1 in Eqs. (9) and (10), respectively

$$\theta_{12}^1 = (12\ \theta_{00}^1 + 4\beta_2\ \theta_{0\infty}^1)/D \tag{15a}$$

$$\theta_{12}^2 = (12\ \theta_{00}^2 + 4\beta_1\ \theta_{\infty 0}^2)/D \tag{15b}$$

Comparing Eqs. (15) and (5), thus

$$C_1 = 12/D;\ C_2 = 4\beta_1/D;\ C_3 = 4\beta_2/D \tag{16}$$

Thus from Eq. (3)

$$C_4 = \beta_1\beta_2/d \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (2), the deflection shape becomes

$$y_{12}(x) = \frac{12 y_{00}(x) + 4(\beta_1 y_{\infty 0}(x) + \beta_2 y_{0\infty}(x)) + \beta_1\beta_2 y_{\infty\infty}(x)}{12 + 4(\beta_1 + \beta_2) + \beta_1\beta_2} \tag{18}$$

Conclusion

A simple expression is derived for the static deflection of a laterally loaded beam with unequal elastic rotational restraints at the ends. The expression given by Eq. (18) is valid for any distribution of the lateral load. It is a function of the end restraint constants and the deflection shapes of the beam under the same loading condition when subjected to the limiting type of supports, i.e., combination of pinned and clamped supports. Deflections corresponding to such conventional support conditions are known and tabulated in handbooks[1] for a variety of practical loading conditions.

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